BU-737-M

ON THE CONSTRUCTION OF MUTUALLY ORTHOGONAL F-HYPERRECTANGLES

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An upper bound on the number of mutually orthogonal F-hyperrectangles with variable numbers of symbols is given. The upper bound is shown to be reached for $N_1 \times N_2 \times \ldots \times N_n$ hyperrectangles where each N_i is a power of s, $i = 1,\ldots,n$, when s is a prime or prime power. A procedure is given for constructing a set of mutually orthogonal F-hyperrectangles from a set of mutually orthogonal latin hypercubes.

AMS 1980 subject classification. 62K05,05B15.

Key words and phrases. Orthogonal arrays, Latin squares, Latin hypercubes, F-squares, F-hyperrectangles, mutually orthogonal F-hyperrectangles.

1. Introduction and definitions.

Hedayat and Seiden (1970) generalized the concept of mutually orthogonal latin squares to the concept of mutually orthogonal F-squares. Cheng (1980) generalized the concept of mutually orthogonal latin hypercubes to the concept of mutually orthogonal F-hyperrectangles. A proof of the universal optimality of a set of mutually orthogonal F-hyperrectangles with variable number of symbols for the elimination of multi-way heterogeneity and as a fractional factorial design was given. Two theorems on the existence of sets of mutually orthogonal F-hyperrectangles with the same number of symbols, a special case, were also given.

In this paper we give an upper bound on the number of mutually orthogonal F-hyperrectangles with variable numbers of symbols. We show that this bound is reached for $N_1 \times N_2 \times \ldots \times N_n$ hyperrectangles where each N_i is a power of s, $i = 1,\ldots,n$, when s is a prime or prime power. We also show how to construct a set of mutually orthogonal F-hyperrectangles from a set of mutually orthogonal latin hypercubes.

Cheng (1980) defined the concept of a F-hyperrectangle as follows. Coordinatize the $\prod_{i=1}^{n} N_i$ cells of an n-dimensional hyperrectangle of size $N_1 \times N_2 \times \ldots \times N_n$ by the n-tuples of integers $(j_1, j_2, \ldots, j_n)$ where $1 \le j_i \le N_i$. An <u>F-hyperrectangle</u> is an arrangement of s symbols into the $\prod_{i=1}^{n} N_i$ cells, where $s \mid \prod_{j \neq i} N_j$ for $i = 1,\ldots,n$, such that each symbol appears $s^{-1}(\prod_{j \neq i} N_j)$ times in each of the N_i sets $H_1^i, H_2^i, \ldots, H_{N_i}^i$, where H_j^i in the set of all cells with j as the ith coordinate, $j = 1,\ldots,N_i$. Two F-hyperrectangles are said to be orthogonal if, when superimposed on one another, every ordered pair of symbols occurs the same number of times.

2. Constructing mutually orthogonal F-hyperrectangles.

Cheng (1980) showed that for a set of m mutually orthogonal F-hyperrectangles of size $N_1 \times N_2 \times \ldots \times N_n$ with the same number of symbols s,

$$m \leq \left(\prod_{i=1}^{n} N_i - \sum_{i=1}^{n} (N_i - 1) - 1\right)/(s - 1) \ .$$

For F-hyperrectangles with differing numbers of symbols we have the following result,

THEOREM 2.1. *If $\{F_1, F_2, \ldots, F_m\}$ is a set of m mutually orthogonal F-hyperrectangles of size $N_1 \times N_2 \times \ldots \times N_n$ and $s_1, s_2, \ldots, s_m$ symbols respectively, then*

$$\sum_{i=1}^{m} (s_i - 1) \leq \prod_{i=1}^{n} N_i - \sum_{i=1}^{n} (N_i - 1) - 1 \ .$$

Proof: From Rao (1973) we have that for an $OA(N; s_1,\ldots,s_n;2)$, the inequality $N - 1 \geq \sum_{i=1}^{n} (s_i - 1)$ holds. Since a set of m mutually orthogonal F-hyperrectangles of size $N_1 \times N_2 \times \ldots \times N_n$ and $s_1, s_2, \ldots, s_m$ symbols is an $OA(\prod_{i=1}^{n} N_i; N_1, N_2, \ldots, N_n, s_1, s_2, \ldots, s_m; 2)$ we have that

$$\prod_{i=1}^{n} N_i - 1 \geq \sum_{i=1}^{n} (N_i - 1) + \sum_{i=1}^{m} (s_i - 1)$$

and so,

$$\sum_{i=1}^{m} (s_i - 1) \leq \prod_{i=1}^{n} N_i - \sum_{i=1}^{n} (N_i - 1) - 1 \quad .$$

Theorem 2.1 therefore gives an upper bound on the number of mutually orthogonal F-hyperrectaugles with differing numbers of symbols. That number we can see depends on the values of the s_i, the number of symbols in each F-hyperrectangle. When the upper bound is reached we have a <u>complete set of mutually orthogonal F-hyperrectangles</u>.

We say that a F-hyperrectangle F <u>decomposes</u> into a set of mutually orthogonal F-hyperrectangles $\{F_1, F_2, \ldots, F_t\}$ if F_i can be obtained from F by identifying different symbols for $i = 1, \ldots, t$. The following theorem deals with the decomposition of F-hyperrectangles where the number of symbols is a prime power.

<u>THEOREM 2.2</u>. <u>If</u> $s = p^h$, <u>where</u> p <u>is a prime or prime power and</u> h <u>is a positive integer, and each</u> N_i <u>is a power of</u> s, $i = 1, \ldots, n$, <u>then a F-hyperrectangle of size</u> $N_1 \times N_2 \times \ldots \times N_n$ <u>and</u> s <u>symbols can be decomposed into</u> $(s - 1)/(p^k - 1)$ <u>mutually orthogonal F-hyperrectangles of size</u> $N_1 \times N_2 \times \ldots \times N_n$ <u>and</u> p^k <u>symbols, for all integers</u> k <u>that divide</u> h .

<u>Proof</u>: Let $s = p^h$, where p is a prime or prime power and h is a positive integer, and let k be an integer that divides h. By Rao (1946) there exists an $(s, (s - 1)/(p^k - 1), p^k, 2)$ orthogonal array. Denote it by the symbol OA. Then OA has the following form:

$$OA = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1s} \\ X_{21} & X_{22} & \cdots & X_{2s} \\ \vdots & & & \\ X_{r1} & X_{r2} & \cdots & X_{rs} \end{pmatrix}$$

where $r = \dfrac{s-1}{p^k - 1}$ and $X_{ij} \varepsilon \{0,1,2,\ldots,p^k - 1\}$ for $i = 1,\ldots,r$ and $j = 1,\ldots,s$. Let F be a F-hyperrectangle of size $N_1 \times N_2 \times \ldots \times N_n$ and s symbols with the symbols denoted as 0, 1, 2, ..., $s - 1$. Replace symbol j of F with symbol $X_{1,(j+1)}$ of OA for $j = 0,1,2,\ldots,s - 1$. It can be seen that the resulting design is a F-hyperrectangle of size $N_1 \times N_2 \times \ldots \times N_n$ and p^k symbols. Call this F-hyperrectangle F_1. Likewise replacing symbol j of F with symbol $X_2,(j+1)$ of OA for $j = 0,1,2,\ldots,s-1$ will give us a second F-hyperrectangle of size $N_1 \times N_2 \times \ldots \times N_n$ and p^k symbols. Call it F_2. Using all r rows of OA we can obtain r F-hyperrectangles $F_1,F_2,\ldots,F_r$ of size $N_1 \times N_2 \times \ldots \times N_n$ and p^k symbols. Since each F_i, $(i = 1,\ldots,r)$ comes from a row of the orthogonal array OA, the set $\{F_1,F_2,\ldots,F_r\}$ forms an orthogonal set of F-hyperrectangles.

Cheng (1980) proved the existence of a complete set of mutually orthogonal F-hyperrectangles of prime power order and the same number of symbols. We have the following result concerning a complete set of mutually orthogonal F-hyperrectangles with differing numbers of symbols.

THEOREM 2.3. *If* $s = p^h$, *where* p *is a prime or prime power and* h *is a positive integer, and each* N_i *is a power of* s, $i = 1,\ldots,n$, *then there exists a complete set* $\{F_1,F_2,\ldots,F_t\}$ *of* t *mutually orthogonal F-hyperrectangles of size* $N_1 \times N_2 \times \ldots \times N_n$ *and* $s_1,s_2,\ldots,s_t$ *symbols respectively, where* $s_i = p^{k_i}$, *for integers* k_i *that divide* h $(i = 1,\ldots,t)$.

Proof: Cheng (1980) showed that if s is a prime power, and each N_i is a power of s, $i = 1,\ldots,n$, then there exists a complete set of $u = (\prod_{i=1}^{n} N_i - \sum_{i=1}^{n} (N_i - 1) - 1)/(s - 1)$ mutually orthogonal F-hyperrectangles of size $N_1 \times N_2 \times \ldots \times N_n$ and s symbols. Let the u orthogonal F-hyperrectangles be labeled as $G_1,G_2,\ldots,G_u$. By theorem 2.2 we can decompose each G_i $(i = 1,\ldots,u)$ into $(s - 1)/(p^{k_i} - 1)$ mutually orthogonal F-hyperrectangles of size $N_1 \times N_2 \times \ldots \times N_n$ and p^{k_i} symbols, where k_i is a positive integer that divides h. Let t be the total number of F-hyperrectangles constructed by the decompositions. Denote these t F-hyperrectangles as $F_1,F_2,\ldots,F_t$. It can be seen that the set $\{F_1,F_2,\ldots,F_t\}$ is an orthogonal set because the F_i's are decompositions of the orthogonal set $\{G_1,G_2,\ldots,G_u\}$.

It remains to show that the orthogonal set $\{F_1,F_2,\ldots,F_t\}$ is complete. From above, F_i has p^{k_i} symbols, where k_i divides h. Now,

$$\begin{aligned}
\sum_{i=1}^{t} (p^{k_i} - 1) &= \sum_{j=1}^{u} [(s - 1)/(p^{k_j} - 1)](p^{k_j} - 1) \\
&= u(s - 1) \\
&= [(\prod_{i=1}^{n} N_i - \sum_{i=1}^{n} (N_i - 1) - 1)/(s - 1)](s - 1) \\
&= \prod_{i=1}^{n} N_i - \sum_{i=1}^{n} (N_i - 1) - 1
\end{aligned}$$

and so the set $\{F_1, F_2, \ldots, F_t\}$ is complete by theorem 2.1.

Cheng (1980) showed that if there exist orthogonal arrays $OA(N_i, n_i, s, 2)$ for $i = 1, \ldots, k$, then there exist m mutually orthogonal F-hyperrectangles of size $N_1 \times N_2 \times \ldots \times N_k$ and s symbols, where $m = \prod_{i=1}^{k} (n_i + 1) - 1 - \sum_{i=1}^{k} n_i$. The following result shows that we can construct more mutually orthogonal F-hyperrectangles if a set of mutually orthogonal k-dimensional latin hypercubes of order s also exists.

THEOREM 2.4. *If there exist orthogonal arrays* $OA(N_i, n_i, s, 2)$ *for* $i = 1, \ldots, k$, *and a set of* t *mutually orthogonal k-dimensional latin hypercubes of order* s, *then there exist* $t \prod_{i=1}^{k} n_i$ *mutually orthogonal F-hyperrectangles of size* $N_1 \times N_2 \times \ldots \times N_k$ *and* s *symbols*.

Proof. Let the s symbols be $0, 1, 2, \ldots, s-1$. Let each $OA(N_i, n_i, s, 2)$ be denoted by A_i. Let $A = A_1 \otimes A_2 \otimes \ldots \otimes A_k$, where $\otimes$ denotes the Kronecker product. Thus A is an $\prod_{i=1}^{k} n_i \times \prod_{i=1}^{k} N_i$ array whose entries are a k-tuple of integers module s. Identify the rows of A by the k-tuples of integers $(i_1, \ldots, i_k)$, $1 \leq i_q \leq n_q$, and the columns by the k-tuples of integers $(j_1, \ldots, j_k)$, $1 \leq j_q \leq N_q$. For each $h = 1, 2, \ldots, k$, the hth coordinate of the element in the $(i_1, \ldots, i_k)$th row and the $(j_1, \ldots, j_k)$th column of A is the element appearing in the i_hth row and the j_hth column of A_h. Thus the $(i_1, \ldots, i_k)$th row of A is obtained from the i_1th row of A_1, the i_2th row of $A_2, \ldots,$ and the i_kth row of A_k.

Let $\{L_1, L_2, \ldots, L_t\}$ be a set of t mutually orthogonal k-dimensional latin hypercubes of order s. Coordinatize the s^k

cells of L_1 by the k-tuples of integers $(\ell_1, \ell_2, \ldots, \ell_k)$ with $0 \le \ell_i \le (s - 1)$. Replacing each entry, $(a_1, a_2, \ldots, a_k)$, with $0 \le a_i \le (s - 1)$, of A with the symbol $0 \le r \le (s - 1)$ coming from the $(a_1, a_2, \ldots, a_k)$ coordinate of L_1, we get an array $\bar{A}$ of the integers $0, 1, 2, \ldots, s - 1$. Using the coordinates of the $\prod_{i=1}^{k} N_i$ columns of $\bar{A}$ as the coordinates of a hyperrectangle of size $N_1 \times N_2 \times \ldots \times N_k$, each of these $\prod_{i=1}^{k} n_i$ rows defines a hyperrectangle of size $N_1 \times N_2 \times \ldots \times N_k$ and s symbols. These $\prod_{i=1}^{k} n_i$ hyperrectangles are F-hyperrectangles that are mutually orthogonal. Using L_2 we can obtain $\prod_{i=1}^{k} n_i$ more F-hyperrectangles that are mutually orthogonal. The $\prod_{i=1}^{k} n_i$ F-hyperrectangles obtained from L_1 are orthogonal to the $\prod_{i=1}^{k} n_i$ hyperrectangles obtained from L_2 since L_1 is orthogonal to L_2. Hence we have $2 \prod_{i=1}^{k} n_i$ mutually orthogonal F-hyperrectangles. Altogether from the set $\{L_1, L_2, \ldots L_t\}$ we can obtain a set of $t \prod_{i=1}^{k} n_i$ mutually orthogonal F-hyperrectangles.

<u>Remark</u>. Theorem 2.4 says that $t \prod_{i=1}^{k} n_i$ mutually orthogonal F-hyperrectangles can be constructed from orthogonal arrays and mutually orthogonal latin hypercubes. Cheng (1980) showed that $\prod_{i=1}^{k} (n_i + 1) - 1 - \sum_{i=1}^{k} n_i$ mutually orthogonal F-hyperrectangles can be constructed from orthogonal arrays. Thus theorem 2.4 can construct more mutually orthogonal F-hyperrectangles if and only if

$$t \prod_{i=1}^{k} n_i > \prod_{i=1}^{k} (n_i + 1) - 1 - \sum_{i=1}^{k} n_i ,$$

that is, if and only if

$$t > \left(\prod_{i=1}^{k} (n_i + 1) - 1 - \sum_{i=1}^{k} n_i\right) / \prod_{i=1}^{k} n_i = u \quad .$$

This gives a lower bound for the number t of mutually orthogonal latin hypercubes needed to obtain more mutually orthogonal F-hyperrectangles than by previous methods. It can be seen that

$$u \leq [1 + \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{k-2}] \prod_{i=1}^{k} n_i / \prod_{i=1}^{k} n_i = 2^k - k - 1 \; .$$

When s is a prime or prime power Kishen (1949) showed that there exists a set of $t = \frac{s^k - 1}{s - 1} - k$ mutually orthogonal k-dimensional latin hypercubes of order s . In this case $t = \frac{s^k - 1}{s - 1} - k \geq 2^k - 1 - k \geq u$.

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